Infra-Red Finite, Physical Electron Propagator in $2 + 1$ Dimensions

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Abstract: In this paper we study the infra-red behaviour of a gauge invariant and physically motivated description of a charged particle in $2 + 1$ dimensions. We show that both the mass shift and the wave function renormalisation are infra-red finite on-shell.

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Introduction

The infra-red (IR) problem is well known to be a consequence of the unjustified neglect of long range asymptotic interactions [1, 2]. This generates logarithmic singularities in 3+1 dimensional QED. However, in 2+1 dimensions naive power counting indicates that these divergences will get worse. In particular, one may now expect additional on-shell linear IR divergences, in quantities where there were previously only logarithms, such as the on-shell wave function renormalisation Z_2, and that some previously IR finite quantities, such as the on-shell mass shift, will pick up logarithmic divergences in $2+1$. Thus $2+1$-dimensional theories offer a tough testing ground for methods used in the familiar $3+1$ case [3, 4].

As well as the automatic attention due to any model field theory, interest in $2+1$ dimensional gauge theories is also based upon their relevance to the high temperature limit and to a range of applications in condensed matter where many systems are described by effective abelian theories where the charged particles are either supposed to describe low-energy electrons or collective excitation, see, e.g., [5–9] and references therein. In such applications it has proven hard to extract physical predictions since the Lagrangian fermion and its Green's functions are gauge dependent. This gauge dependence shows that the fermion cannot be interpreted as a physical particle. Indeed this is directly related to the IR problem since, even at asymptotically large times, the Lagrangian fermion does not become a free particle but experiences a residual interaction and is not gauge invariant [1, 10].

The aim of this paper is to apply a systematic construction of gauge invariant, physical variables to $2+1$ dimensional electrodynamics. This is set up in such a way that the physical fields do, at large times, have a particle interpretation. In particular we will show that the on-shell mass shift and wave function renormalisation constants of the physical propagator are IR finite. This we will demonstrate in both spinor and scalar QED, since in $2+1$ dimensions the IR structure depends upon the spin of massive charged particles. This work builds upon a series of papers [11,12] investigating the $3+1$ dimensional theory, and so we should very briefly recall the motivation and results of that work. After that we will study the physical propagator in $2+1$ dimensions before presenting some conclusions.

What is an Electron?

As a physical particle any description of an electron must be gauge invariant, however, for the matter field we have $\psi \to U^{-1}\psi$. This leads to the gauge invariant ansatz, $h^{-1}\psi$, where the field dependent dressing, h^{-1} , must transform [11,13] as: $h^{-1} \to h^{-1}U$. There are, of course, many solutions to this minimal requirement of gauge invariance and to single out the physical description of a charged particle propagating with four velocity $u^\mu = \gamma(\eta + v)$ (where γ is the usual Lorentz factor, η is $(1, 0, 0, 0)$ and $v = (0, \boldsymbol{v})$ is the

velocity) we introduce an additional dressing equation $u \cdot \partial h^{-1}(x) = -ieh^{-1}(x) u \cdot A(x)$. This equation can be motivated either from a study of the heavy charge effective theory or from the form of the asymptotic interaction Hamiltonian.

In QED these two requirements lead to a solution where the dressing factors into two parts: a minimal structure (χ) which ensures the gauge invariance of the dressed charge and an extra, separately gauge invariant factor (K) which is needed to fulfill the dressing equation.

$$h^{-1}\psi = \mathrm{e}^{-ieK(x)} \mathrm{e}^{-ie\chi(x)} \,, \tag{1}$$

where the minimal term in the dressing is given by

$$\chi(x) = \frac{\mathcal{G} \cdot A}{\mathcal{G} \cdot \partial} \,, \tag{2}$$

with $\mathcal{G}^\mu = (\eta + v)^\mu (\eta - v) \cdot \partial - \partial^\mu$, and the extra (gauge invariant) structure is

$$K(x) = \int_{\pm\infty}^{x^0} (\eta + v)^\mu \frac{\partial^\nu F_{\nu\mu}}{\mathcal{G} \cdot \partial} (x(s)) \, ds \,. \tag{3}$$

In this last expression the integral is along the world-line of a massive particle with four-velocity u^μ. A more detailed explanation of the origin of the dressing can be found in [12] and [14].

It has been shown in $3+1$ dimensions that these dressed fields asymptotically yield a particle description [10]. It has been shown in explicit calculations that the on-shell propagator of this description of a charged particle is IR finite in $3+1$ dimensions. It is essential for this cancelation that the point where one goes on-shell corresponds to the velocity parameter in the dressing. Note that this has been demonstrated for both fermionic and scalar matter [15,16], although we recall that the IR divergences of QED in $3+1$ are spin independent as long as the matter fields are massive.

Below we will directly carry over the dressed field of $3+1$ dimensions to the lower dimensional case, i.e., we sum spatial indices from 1 to 2 instead of 1 to 3. It has previously been shown that this construction generates the static inter-quark potential $V(r) = \ln(r)$ and that the minimal part of the dressing, (χ), needed for gauge invariance, produces the anti-screening part of the potential in $2+1$ dimensions [17].

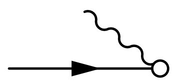
The Dressed Propagator in $2+1$ Dimensions

To analyse the physical electron propagator in $2+1$ dimensional scalar QED, we will extract the IR divergences in each of the different sorts of diagrams contributing to the dressed electron propagator in scalar QED. We then repeat this, very briefly, in fermionic QED to investigate the spin dependence of the IR divergences associated with the propagator.

In scalar QED our physical field is given by

$$h^{-1}(x)\phi(x) = e^{-ieK(x)}e^{-ie\chi(x)}\phi(x)\,. \tag{4}$$

The Feynman rules for the dressed Green's functions are the usual ones with the addition of two new rules corresponding to the dressings as shown in Figure 1.


$$=\frac{eV^\mu}{V\cdot k}$$

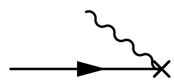

$$=\frac{eW^\mu}{V\cdot k}$$

Figure 1: *The Feynman rules from expanding the dressing.*

The first vertex comes from the minimal (χ) part of the dressing, and the second corresponds to the additional, separately gauge invariant (K) dressing. Here V and W are defined as follows:

$$V^\mu := k\cdot(\eta+v)\,(\eta+v)^\mu - k^\mu\,; \quad W_\mu = \frac{k\cdot(\eta+v)\,k_\mu - k^2\,(\eta+v)_\mu}{k\cdot\eta}, \tag{5}$$

where v is the velocity of the on-shell particle with momentum $p = m\gamma(\eta+v)$, and k is the incoming momentum of the photon. Note that $W\cdot k = 0$ as a consequence of the gauge invariance of that part of the dressing.

We draw all the possible one loop diagrams for the electron propagator when we include the above dressing and then analyse their IR structure diagram by diagram. Since the dressed fields are by construction gauge invariant, the sum of these structures must be gauge invariant. Finally, the cancelation of on-shell IR divergences will be shown explicitly. Our procedure is an extension of [18, 19] which is needed to treat the richer IR structure of the 2+1 dimensional theory.

The relevant diagrams are shown in Figure 2. These include both the minimal and additional dressings, together with all the massless tadpoles shown in Figure 3. The usual on-shell propagator, given by the sum of Figure 2(a) and 2(b), has by power counting both logarithmic (in both δm and Z_2) and also linear IR divergences (in Z_2). The remaining diagrams, 2(c) – 2(j), come from expanding both parts of the dressing, where 2(c) – 2(e) involve the perturbative expansion of the minimal (χ) part of the dressing (see also Section 3 of [19]); 2(f) and 2(g) are cross terms from expanding both dressing structures and the diagrams 2(h) – 2(j) come from expanding the additional (K) term.

The contribution of the usual covariant diagram 2(b) to the propagator has the form

$$iS^{2b}(p) = \frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{(2p-k)^\mu(2p-k)^\nu}{(p-k)^2-m^2}\,. \tag{6}$$

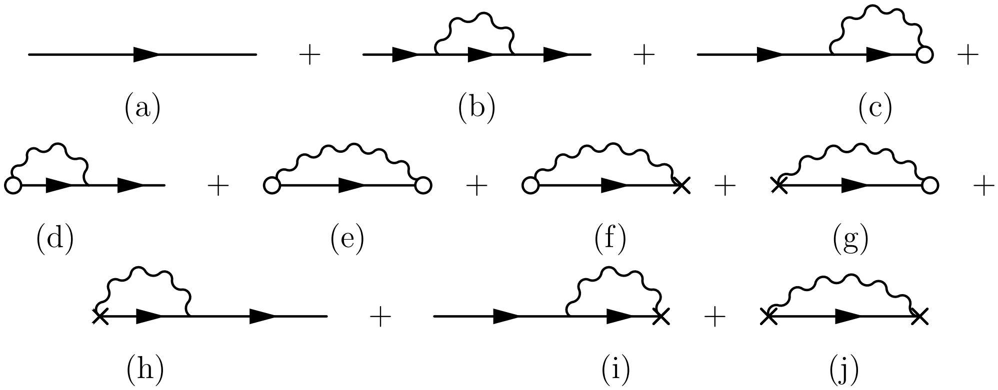


Figure 2: *The one-loop Feynman diagrams in the electron propagator which contain IR-divergences when both the minimal and extra dressing are included.*

Note that, to bring out the gauge invariance of our final result, we do not specify the form of the photon propagator, $D_{\mu\nu}$. Our procedure is to extract the IR divergences from each diagram for both double and single pole structures.

This diagram has an on-shell IR divergence which, as is well known, causes δm (the mass renormalisation constant) to be IR divergent in 2+1. In the case of 3+1 dimensions we find similar IR infinities (but there only in Z_2) by extracting a power of (p^2-m^2) [19]. The formal procedure is to Taylor expand about $p^2 = m^2$. After dropping the IR finite term, we obtain from diagram 2(b) the following IR divergent contributions to the mass shift (double pole) and the wave function renormalisation constant (single pole):

$$
\begin{aligned}
iS^{2b}(p) \;=\; & -\frac{2e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{p^\mu p^\nu}{(p\cdot k}\\
& +\frac{e^2}{(p^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\left\{\left[-\frac{p^\mu p^\nu}{(p\cdot k)^2}\right]\right.\\
& \left.+\left[-\frac{1}{m^2}\frac{p^\mu p^\nu}{p\cdot k}+\frac{p^\mu k^\nu}{2(p\cdot k)^2}+\frac{p^\nu k^\mu}{2(p\cdot k)^2}-\frac{p^\mu p^\nu}{(p\cdot k)^2}\frac{k^2}{p\cdot k}\right]\right\}. \qquad (7)
\end{aligned}
$$

As expected from power counting, there are only logarithmic divergences in δm but both linear and logarithmic ones in Z_2.

From diagram 2(c), the Feynman rules yield

$$
iS^{2c}(p) = \frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu}{V\cdot k}\frac{(2p-k)^\nu}{(p-k)^2-m^2}. \qquad (8)
$$

Simple power counting tells us that the term proportional to p has an off-shell IR divergence which is not well defined. In order to make it well defined we use the identity

(see also [14]).

$$\frac{1}{(p-k)^2-m^2}=\frac{1}{p^2-m^2}\left[1+\frac{2p\cdot k-k^2}{(p-k)^2-m^2}\right]. \qquad (9)$$

This produces a double pole structure. Using a Taylor expansion to find the single pole structures we obtain the following contribution to the diagram 2(c):

$$\begin{aligned} iS^{2c}(p) \;=\; & -\frac{2e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{p^\mu V^\nu}{V\cdot k} \\ & +\frac{e^2}{(p^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\left\{\left[-\frac{p^\mu V^\nu}{p\cdot k\,V\cdot k}\right]\right. \\ & \left.\quad +\left[-\frac{1}{m^2}\frac{p^\mu V^\nu}{V\cdot k}-\frac{V^\mu k^\nu}{2p\cdot k\,V\cdot k}+\frac{p^\mu V^\nu}{2p\cdot k\,V\cdot k}\frac{k^2}{p\cdot k}\right]\right\}. \end{aligned} \qquad (10)$$

Here we have dropped the 1 in the square bracket of (9), since it is a double pole massless tadpole and corresponds to an *odd* k integral. In *any* reasonable regulator such terms must vanish. The contribution of diagram 2(d) is easily seen to be identical to this. The contribution of diagrams 2(h) and 2(i) to the propagator can now be immediately obtained by changing all the V-factors to W's in (10).

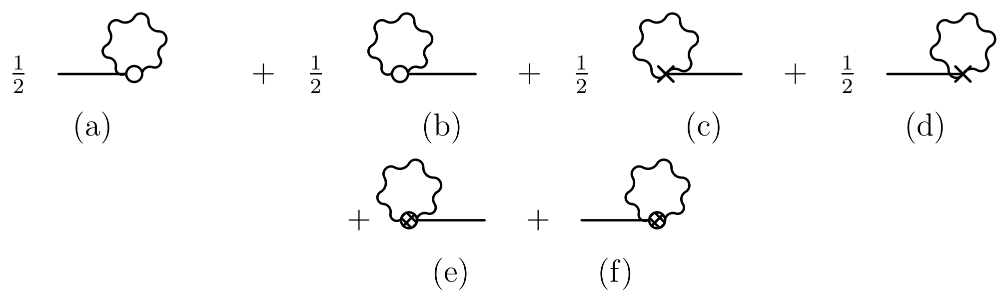


Figure 3: *All these one loop massless tadpoles will cancel during the process of extracting IR divergences from diagrams 2e - 2g and 2j. The hatched circle vertex indicates the generic contributions of both parts of the dressing.*

The off-shell divergences in the rainbow diagrams 2(e) – 2(g) and 2(j), are again worse in the lower dimensional case. To define them we now use (9) twice. We so find that each diagram contains a double pole IR infinity. We also need to perform a Taylor expansion to extract any single pole structures. To see this explicitly we calculate diagram 2(e) whose contribution to the propagator is

$$iS^{2e}(p)=e^2\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}\frac{1}{(p-k)^2-m^2}. \qquad (11)$$

This diagram has off-shell IR divergences and we make use of (9) to rewrite it as

$$iS^{2e}(p) = \frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}\left[1+\frac{2p\cdot k-k^2}{(p-k)^2-m^2}\right]. \tag{12}$$

The first term in the square bracket is a single pole massless tadpole which cancels the diagrams 3(a) and 3(b). When the remaining rainbow diagrams are calculated, all other diagrams in Figure 3 are similarly canceled. By power counting we can see that the third term in the square bracket $(-k^2)$ of (12) is well defined, but it is IR divergent on-shell. The second term $(2p\cdot k)$ still has an off-shell IR divergence. We use the identity (9) again and obtain

$$\begin{aligned} iS^{2e}(p) \quad = \quad & \frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}2p\cdot k\left[1+\frac{2p\cdot k-k^2}{(p-k)^2-m^2}\right] \\ & +\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}\frac{k^2}{2p\cdot k}. \end{aligned} \tag{13}$$

Again an *odd* double pole massless tadpole is dropped. All the other integrals are now well defined. We now go on-shell and drop all IR finite terms to establish the following divergent contribution to the diagram 2(e):

$$\begin{aligned} iS^{2e}(p) \quad = \quad & \frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}p\cdot k \\ & -\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}\left[1+\frac{p\cdot k}{m^2}\right]. \end{aligned} \tag{14}$$

The contribution of the rainbow diagram 2(j) to the propagator can now be easily obtained by changing all the V-factors to W's in (14). We change one V to W for the diagrams 2(f) and 2(g).

Note that we have both logarithmic and linear divergences for the single pole, but only a logarithmically divergent structure for the double pole. This is in accord with power counting and explicit perturbative calculations of the (non-dressed) propagator [20].

We now combine all these results to obtain the various gauge invariant structures in the dressed electron propagator. All the IR divergent terms in the *double pole* can be written in the following gauge invariant form:

$$\begin{aligned} -\frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3} \quad & \left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right. \\ & \left.\times\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\right\}2p\cdot k. \end{aligned} \tag{15}$$

This form displays the gauge invariance of the dressed Green's functions: any modification of the Feynman gauge photon propagator will introduce either a k_μ or k_ν factor, but these additional structures will vanish on multiplying these into the square bracket in the above structure. We note here that we have dropped double pole *odd* massless tadpoles, which are not, themselves, separately gauge invariant but which must vanish in any reasonable regularisation scheme.

In the single pole terms we have structures which are not only logarithmically but also linearly divergent. Putting together the *linear divergences* that arise from the *single pole*, one finds the gauge invariant structure:

$$-\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\right\}. \tag{16}$$

This is similar to (15) (up to the factor $2p\cdot k$) and is similarly gauge invariant.

Finally the sum of the *logarithmically divergent terms in the single pole* has the gauge invariant form:

$$\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\left\{\left[\frac{k^\nu}{p\cdot k}-\frac{p^\nu}{p\cdot k}\frac{k^2}{p\cdot k}\right]\right.$$
$$\left.-\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\frac{p\cdot k}{m^2}\right\}. \tag{17}$$

To show that these IR divergences cancel at the correct point on the mass shell, it is useful to consider the linear combination:

$$\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}. \tag{18}$$

Using the definitions of V^μ and W^μ given above, we observe that this combination adds to zero at the correct point on the mass shell, i.e.,

$$\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}=0, \quad \text{if} \quad p^\mu=m\gamma(\eta+v)^\mu. \tag{19}$$

This can be seen to be a consequence of the *dressing equation* since expanding $u\cdot\partial h^{-1}=-ieh^{-1}u\cdot A$, in e, and demanding $p^\mu=u^\mu$, the correct point on the mass shell, we obtain (19). Applying this to (15), (16) and (17) we find that the on-shell mass and wave function renormalisation constants are both IR finite. (As in 3+1 dimensions, this cancelation only occurs if the dressing parameter v and on-shell point correspond to each other via $p^\mu=mu^\mu=m\gamma(\eta+v)^\mu$.)

Having shown the cancelation of the various IR divergences that occur in the dressed scalar electron propagator in 2+1 dimensions, we briefly sketch the results of parallel calculations in fermionic QED.

The double pole gauge invariant structure, (15), is identical in the fermionic theory if $1/(p^2 - m^2)$ is replaced by $1/(\not{p} - m)$, confirming the spin independence of the IR divergences in mass renormalisation in 2+1 dimensions. As in scalar theory there are *odd* massless tadpoles which are not separately gauge invariant but must vanish.

The single pole linear IR structures (16) are also identical in both theories, the leading IR singularities being spin independent as one would expect. However, for the sub-leading divergences in fermionic QED we now find the spin dependent structure:

$$e^2 \int \frac{d^3k}{(2\pi)^3} \left[\frac{p^\mu}{p \cdot k} - \frac{V^\mu}{V \cdot k} - \frac{W^\mu}{V \cdot k} \right] D_{\mu\nu} \left[\frac{k^\nu}{p \cdot k} - \frac{p^\nu}{p \cdot k} \frac{k^2}{p \cdot k} \right] \frac{1}{\not{p} - m} \,. \tag{20}$$

Nevertheless from (19), we can immediately show that the electron propagator in fermionic QED is also IR finite.

Conclusions

In this paper we have studied the properties of the on-shell electron propagator in 2+1 dimensions, where the IR divergence structures are far richer than in 3+1 dimensions. We have shown that if we use the full dressing to solve the dressing equation, then both the mass shift and the wave function renormalisation constant are IR finite, despite there now being both linear and logarithmic IR structures. The different IR structures cancel separately at the correct point on the mass shell. These results were established in both fermionic and scalar QED.

We have thus demonstrated that, as in $3 + 1$ dimensions [19], the dressed theory gives an IR finite description of charged particles propagating on-shell. The dressing is able to deal with the significantly more complex IR structures in this lower dimensional theory at one loop. This description, we argue, should now be applied to the effective abelian theories of condensed matter systems [5–9] where a gauge-invariant description of on-shell charge propagation is needed.

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